$x = a + b \cdot \sqrt{2} \quad ; \quad a, b \in \mathbb{Q}$

$\oplus$: $\quad \mathbb{1} = 0 + 0 \cdot \sqrt{2} \Rightarrow 0 \in \mathbb{Q}$

$\tilde{x} = -a + (-b) \cdot \sqrt{2} \Rightarrow -a, -b \in \mathbb{Q}$ $\Big\}$ abelsche Gruppe

$\circledast$: $\quad \mathbb{1} = 1 + 0 \cdot \sqrt{2} \Rightarrow 1, 0 \in \mathbb{Q}$ → abelsches Monoid

$\bar{x}$: $\quad x * \bar{x} = \mathbb{1}$

$$(a + b \cdot \sqrt{2}) * (\bar{a} + \bar{b} \cdot \sqrt{2}) = 1 + 0 \cdot \sqrt{2}$$

$$a \cdot \bar{a} + a\bar{b}\sqrt{2} + \bar{a}b\sqrt{2} + 2b\bar{b} = 1 + 0\sqrt{2}$$

$$(a \cdot \bar{a} + 2b\bar{b}) + (a\bar{b} + \bar{a}b) \cdot \sqrt{2} = 1 + 0\sqrt{2}$$

$$\left| \begin{array}{l} a \cdot \bar{a} + 2b \cdot \bar{b} = 1 \\ a \cdot \bar{b} + \bar{a} \cdot b = 0 \end{array} \right|$$

$$\left|\begin{aligned} a\cdot\boxed{\bar a} + 2b\cdot\bar b &= 1 \\ a\cdot\bar b + \bar a\cdot b &= 0 \end{aligned}\right| \Rightarrow \bar a = \boxed{-\frac{a\bar b}{b}}$$

$$a\left(-\frac{a\bar b}{b}\right) + 2b\cdot\bar b = 1$$

$$\left(-\frac{a^2\bar b}{b}\right) + 2b\bar b = 1 \qquad |\cdot b$$

$$-a^2\bar b + 2b^2\bar b = \bar b(-a^2+2b^2) = b$$

$$\bar b = \boxed{\frac{b}{2b^2-a^2}}$$

$$\bar a = -\frac{a}{b}\cdot\frac{b}{2b^2-a^2} = \frac{-a}{2b^2-a^2} = \frac{a}{a^2-2b^2}$$

$$\bar x = \frac{a}{a^2-2b^2} + \frac{b}{2b^2-a^2}\cdot\sqrt{2}$$

Probe:

$$(a+b\sqrt{2})\cdot\left(\boxed{\frac{a}{a^2-2b^2}}+\frac{b}{2b^2-a^2}\sqrt{2}\right)=1+0\sqrt{2}$$

$$\frac{a^2}{a^2-2b^2}+\frac{ab}{2b^2-a^2}\cdot\sqrt{2}+\frac{ba}{a^2-2b^2}\cdot\sqrt{2}+\frac{2b^2}{2b^2-a^2}$$

$$\left(\frac{a^2}{a^2-2b^2}+\frac{2b^2}{2b^2-a^2}\right)+\left(\frac{ab}{2b^2-a^2}+\frac{ab}{a^2-2b^2}\right)\cdot\sqrt{2}$$

$-(-a^2+2b^2)$

$$\left[\underbrace{\frac{a^2}{a^2-2b^2}-\frac{2b^2}{a^2-2b^2}}_{1}\right]+\left[\underbrace{\frac{ab}{2b^2-a^2}-\frac{ab}{2b^2-a^2}}_{0}\right]\cdot\sqrt{2}$$

$$1+0\cdot\sqrt{2}=1$$

Prüfe $a^2-2b^2=0 \Rightarrow a=b\cdot\sqrt{2}$ nur bei $a=b=0$

$x=0+0\cdot\sqrt{2}=1$ $\oplus$

Distributiv:

$$(x_1 + x_2) \cdot x_3 = \boxed{(x_1 \cdot x_3) + (x_2 \cdot x_3)}$$

$$\left[(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2})\right] \cdot (a_3 + b_3\sqrt{2})$$

$$\left[(a_1 + a_2) + (b_1 + b_2) \cdot \sqrt{2}\right] \cdot (a_3 + b_3\sqrt{2})$$

$$\left[\underline{a_3 \cdot (a_1 + a_2)} + \underline{2b_3 \cdot (b_1 + b_2)}\right] +$$

$$\left[a_3 \cdot (b_1 + b_2) + b_3 \cdot (a_1 + a_2)\right] \sqrt{2}$$

$$\left[(a_1 + b_1\sqrt{2}) \cdot (a_3 + b_3\sqrt{2})\right] + \left[(a_2 + b_2\sqrt{2}) \cdot (a_3 + b_3\sqrt{2})\right]$$

$$\left[(\underline{a_1 a_3} + \underline{2b_1 b_3}) + (a_3 b_1 + a_1 b_3)\sqrt{2}\right] +$$

$$\left[(\underline{a_2 a_3} + \underline{2b_2 b_3}) + (a_2 b_3 + b_2 a_3)\sqrt{2}\right]$$