$$x = a + b \cdot \sqrt{2} \quad ; \quad a; b \in \mathbb{Q}$$

$$+ : \quad \mathbb{1} = 0 + 0 \cdot \sqrt{2} \quad ; \quad 0 \in \mathbb{Q}$$

$$\bar{x} = -a + (-b) \cdot \sqrt{2} \quad ; \quad -a; -b \in \mathbb{Q}$$

abel. Gruppe

$$* : \quad \mathbb{1} = 1 + 0 \cdot \sqrt{2} \quad ; \quad 1; 0 \in \mathbb{Q}$$

abel. Monoid

$$x * \bar{x} = \mathbb{1}$$

$$(a + b \cdot \sqrt{2}) * (\bar{a} + \bar{b}\sqrt{2}) = 1 + 0 \cdot \sqrt{2}$$

$$a \cdot \bar{a} + a \cdot \bar{b}\sqrt{2} + b\sqrt{2} \cdot \bar{a} + b\underline{\sqrt{2}} \cdot \bar{b}\underline{\sqrt{2}}$$

$$\underbrace{(a \cdot \bar{a} + 2b \cdot \bar{b})}_{1} + \underbrace{(a \cdot \bar{b} + b \cdot \bar{a})}_{0} \sqrt{2}$$

$$1$$

$$\left| \begin{array}{l} a \cdot \bar{a} + 2b \boxed{\bar{b}} = 1 \\ a\bar{b} + b \cdot \bar{a} = 0 \end{array} \right| \quad \Rightarrow \boxed{\bar{b}} = -\frac{\boxed{\bar{a}} \cdot b}{a}$$

$$a \cdot \bar{a} + 2b \cdot \left(-\frac{\bar{a} \cdot b}{a} \right) = 1$$

$$a \cdot \bar{a} - \frac{2b^2 \bar{a}}{a} = 1 \qquad \Big| \cdot a$$

$$a^2 \cdot \bar{a} - 2b^2 \bar{a} = \bar{a} \cdot (a^2 - 2b^2) = a \qquad \Big| : (a^2 - 2b^2)$$

$$\bar{a} = \boxed{\frac{a}{a^2 - 2b^2}}$$

$$\bar{b} = -\frac{\cancel{a}}{a^2 - 2b^2} \cdot \frac{b}{\cancel{a}} = \frac{b}{2b^2 - a^2}$$

$$\bar{x} = \frac{a}{a^2 - 2b^2} + \frac{b}{2b^2 - a^2} \cdot \sqrt{2}$$

Probe: $\quad x * \bar{x} = 1 \quad : \quad (a + b\cdot\sqrt{2}) * \left(\frac{a}{a^2-2b^2} + \frac{b}{2b^2-a^2}\cdot\sqrt{2}\right)$

$$\frac{a^2}{a^2-2b^2} + \frac{ab}{2b^2-a^2}\cdot\sqrt{2} + \frac{b\cdot a}{a^2-2b^2}\cdot\sqrt{2} + \frac{2b^2}{\underset{-a^2+2b^2}{2b^2-a^2}}$$

$$\underbrace{\left(\frac{a^2}{a^2-2b^2} - \frac{2b^2}{a^2-2b^2}\right)}_{\frac{a^2-2b^2}{a^2-2b^2}=1} + \underbrace{\left(\frac{ab}{2b^2-a^2} - \frac{ab}{2b^2-a^2}\right)}_{0}\sqrt{2} = 1$$

$$\bar{a} = \frac{a^2}{a^2-2b^2} \longrightarrow a^2-2b^2 = 0 \quad 1+2b^2$$

$$a = b\cdot\sqrt{2}$$

$$a = b = 0 \implies 0 + 0\cdot\sqrt{2} = 1$$

Distributivgesetz.

$$x_1 * (x_2 + x_3) = (x_1 * x_2) + (x_1 * x_3)$$

$$(a_1 + b_1 \cdot \sqrt{2})\left[(a_2 + b_2\sqrt{2}) + (a_3 + b_3\sqrt{2})\right]$$

$$(a_1 + b_1\sqrt{2})\left[(a_2 + a_3) + (b_2 + b_3)\sqrt{2}\right]$$

$$\left[a_1 \cdot (a_2 + a_3) + 2b_1 \cdot (b_2 + b_3)\right] + \left[b_1 \cdot (a_2 + a_3) + a_1 \cdot (b_2 + b_3)\right]\sqrt{2}$$

$$(a_1 + b_1\sqrt{2})(a_2 + b_2\sqrt{2}) + (a_1 + b_1\sqrt{2}) \cdot (a_3 + b_3\sqrt{2})$$

$$\left[(a_1 a_2 + 2b_1 b_2) + (a_1 b_2 + a_2 b_1) \cdot \sqrt{2}\right] +$$

$$\left[(a_1 a_3 + 2b_1 b_3) + (a_1 b_3 + a_3 b_1)\sqrt{2}\right]$$