10) $$\sum_{k=2}^{n} \underbrace{(k-1)\cdot \ln\left(\frac{k}{k-1}\right)}_{a_k} = \underbrace{n\cdot \ln(n) - \ln(n!)}_{S_n}$$

$$a_{n+1} = [(n+1)-1]\cdot \ln \frac{n+1}{(n+1)-1} = n\cdot \ln\left(\frac{n+1}{n}\right)$$

$$S_{n+1} = (n+1)\cdot \ln(n+1) - \ln((n+1)!)$$

$n=2$: $$a_2 = S_2$$

$$(2-1)\cdot \ln\left(\frac{2}{2-1}\right) = 2\cdot \ln(2) - \ln(2!)$$

$$\ln(2) = \underbrace{2\cdot \ln(2)}_{\ln 2} - \ln(2) \quad \checkmark$$

$$= \ln 2^2 - \ln 2$$

$$= \ln \frac{2^2}{2} = \ln \frac{4}{2} = \ln 2 \quad \checkmark$$

$$S_n + a_{n+1} = S_{n+1}$$

$$\Leftrightarrow \quad n \cdot \ln(n) - \ln(n!) + n \cdot \ln\left(\frac{n+1}{n}\right) =$$

$$(n+1) \cdot \ln(n+1) - \ln((n+1)!)$$

$$\Leftrightarrow \quad n \cdot \ln(n) - \ln(n!) + n \cdot \left[\ln(n+1) - \ln(n)\right] =$$

$$n \cdot \ln(n+1) + 1 \cdot \ln(n+1) - \ln\left[(n+1) \cdot n!\right]$$

$$- \left[\ln(n+1) + \ln(n!)\right]$$

$$\Leftrightarrow \quad n \cdot \ln(n) - \ln(n!) + n \cdot \ln(n+1) - n \cdot \ln(n) =$$

$$n \cdot \ln(n+1) + \ln(n+1) - \ln(n+1) - \ln(n!)$$

$$-\ln(n!) + n \cdot \ln(n+1) = n \cdot \ln(n+1) - \ln(n!) \Leftrightarrow 0 = 0$$

15)

a) $\sum_{k=3}^{9}\left(\frac{1}{2}\right)^k - \sum_{k=3}^{\infty}\left(\frac{2}{3k}\right)^2 = \left(\frac{1}{4}-\frac{1}{512}\right)-\left(\frac{2}{27}\pi^2-\frac{5}{9}\right)$

$$\sum_{k=0}^{6}\left(\frac{1}{2}\right)^{k+3} = \frac{1}{8}\cdot\sum_{k=0}^{6}\left(\frac{1}{2}\right)^k = \frac{1}{8}\cdot\frac{1-\left(\frac{1}{2}\right)^7}{1-\frac{1}{2}} = \frac{1}{4}\cdot\left(1-\frac{1}{128}\right)$$

$$\sum_{k=3}^{\infty}\left(\frac{2}{3k}\right)^2 = \sum_{k=3}^{\infty}\frac{4}{9k^2} = \frac{4}{9}\cdot\sum_{k=3}^{\infty}\frac{1}{k^2}$$

$\rightarrow \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$

$$\Rightarrow \frac{4}{9}\cdot\left[\frac{\pi^2}{6}-\left(\frac{1}{1^2}+\frac{1}{2^2}\right)\right]$$

$$\frac{4}{9}\cdot\left(\frac{\pi^2}{6}-\frac{5}{4}\right) = \frac{2}{27}\pi^2-\frac{5}{9}$$