Klausur 2016/17

Nr. 4 $\quad 1+4+7+\dots+(3n-2) = \frac{1}{2}\cdot n\cdot(3n-1)\ ;\ n\geq 1$

1. 2. 3. n.

$$\sum_{k=1}^{n} \underbrace{(3k-2)}_{a_k} = \underbrace{\tfrac{1}{2}n\cdot(3n-1)}_{S_n}$$

$n=1$: $\quad a_1 = S_1$

$$3\cdot 1-2 = 1 = \tfrac{1}{2}\cdot 1\cdot(3\cdot 1-1) = \tfrac{1}{2}\cdot 2 \quad ✓$$

Prämisse: Es gilt $\boxed{\sum_{k=1}^{n}(3k-2) = \frac{1}{2}n(3n-1)}$ für $n\geq 1$

$\underline{n+1}$:

$$\sum_{k=1}^{n+1}(3k-2) = \sum_{k=1}^{n}(3k-2) + a_{n+1}$$

$$S_{n+1} = S_n + a_{n+1}$$

$$\frac{1}{2}\cdot(n+1)\cdot[3(n+1)-1) = \frac{1}{2}n\cdot(3n-1) + (3\cdot(n+1)-2) \quad |\cdot 2$$

$$(n+1)\cdot(3n+2) = n\cdot(3n-1) + 2\cdot(3n+1)$$

$$3n^2 + \underbrace{3n + 2n}_{5n} + 2 = 3n^2 \underbrace{- n + 6n}_{5n} + 2 \quad | -3n^2 - 5n - 2$$

$$0 = 0 \quad ✓$$

8) a) $\frac{1}{2}\cdot\sum_{k=3}^{\infty}\frac{2^{k+2}}{k!} = \frac{1}{2}\cdot\sum_{k=3}^{\infty}\frac{2^{k}\cdot 2^{2}}{k!} = 2\cdot\sum_{k=3}^{\infty}\frac{2^{k}}{k!}$

$\sum_{k=0}^{\infty}\frac{x^{k}}{k!} = e^{x}$

$$2\cdot\left[\underbrace{\sum_{k=0}^{\infty} 2^{k}/k!}_{e^{2}} - \underbrace{\left(\frac{2^{0}}{0!}+\frac{2^{1}}{1!}+\frac{2^{2}}{2!}\right)}_{5}\right]$$

$$2\cdot\left(e^{2} - 5\right) = 2e^{2}-10$$

$$5)\ 1024 \sum_{k=1}^{4} (1/2)^{2\cdot(2+k)} = 1024 \cdot \sum_{k=1}^{4} (1/2)^{4} \cdot (1/2)^{2k}$$

$$= \frac{2^{10}}{2^{4}} = 2^{6} \cdot \sum_{k=1}^{4} (1/4)^{k}$$

$$\sum_{k=0}^{n} q^{k} = \frac{1-q^{n+1}}{1-q}$$

$$2^{6} \cdot \sum_{k=0}^{4} (1/4)^{k+1} = 2^{6} \cdot 1/4 \cdot \sum_{k=0}^{4} (1/4)^{k}$$

$$2^{4} \cdot \frac{1-(1/4)^{5}}{1-1/4} = \frac{2^{6}}{3} \cdot \left(1-(1/4)^{5}\right)$$

$$= 64/3 \cdot \left(1 - 1/1024\right)$$

7) $a_{n+1} = \sqrt{6 + 5 \cdot a_n}$; $a_1 = 2$; $n \geq 1$

$a_2 = \sqrt{6 + 5 \cdot a_1} = \sqrt{6 + 5 \cdot 2} = 4$

$a_3 = \sqrt{6 + 5 \cdot a_2} = \sqrt{6 + 5 \cdot 4} = \sqrt{26} \approx 5$

Behauptung $a_{n+1} > a_n$

$n = 1$ $a_2 > a_1 \iff 4 > 2$ ✓

$n+1$: $a_{n+2} > a_{n+1}$

$\sqrt{6 + 5 a_{n+1}} > \sqrt{6 + 5 \cdot a_n}$ $\quad | \uparrow^2$

$6 + 5 a_{n+1} > 6 + 5 a_n$ $\quad | -6 \quad \cdot \frac{1}{5}$

$a_{n+1} > a_n$

$\sqrt{6+5\cdot a_n}$

Da die Folge streng monoton steigt, muss $a_1 = 2$ untere Schranke sein.

Behauptung $\quad a_n < 6 \quad$ (obere Schranke)

$n = 1 \qquad a_1 < 6 \qquad a_1 = 2 < 6 \quad ✓$

$n+1: \qquad a_n < 6 \quad | \cdot 5$

$$5 \cdot a_n < 6 \cdot 5 = 30 \quad | +6$$

$$6 + 5 \cdot a_n < 30 + 6 = 36 \quad | \sqrt{}$$

$$\underbrace{\sqrt{6 + 5 \cdot a_n}} < \sqrt{36}$$

$$a_{n+1} < 6$$

Grenzwert

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty} a_n = r$$

$$\sqrt{6+5\cdot r} = r \quad |\uparrow^2$$

$$6+5r = r^2 \quad |-6-5r$$

$$r^2-5r-6=0$$

$$(r-6)(r+1)=0$$

$$r_1=6 \lor r_2=-1$$

$\mathbb{L}=\{6\}$, da $a_1=2$ untere Schranke ist.