6) $10^n + 3\cdot 4^{n+2} + 5$; $n \geq 0$ ist durch 9 teilbar.

$n=0$ $\quad 10^0 + 3\cdot 4^{0+2} + 5 = 1 + 48 + 5 = 54 = 9\cdot \underset{\in \mathbb{Z}}{6}$ ✓

Prämisse: $\boxed{10^n + 3\cdot 4^{n+2} + 5} = 9\cdot k$; $n \geq 0$

$k \in \mathbb{Z}$

$n+1 \quad 10^{n+1} + 3\cdot 4^{n+3} + 5$

$3\cdot 4^{n+2+1}$

$3\cdot 4^{n+2}\cdot 4^1$

$10\cdot 10^n + 4\cdot(3\cdot 4^{n+2}) + 5$

$(10^n + (3\cdot 4^{n+2}) + 5) + 9\cdot 10^n + 3\cdot(3\cdot 4^{n+2})$

$\underbrace{(10^n + 3\cdot 4^{n+2} + 5)}_{9\cdot k_1} + 9\cdot 10^n + 9\cdot 4^{n+2}$

$+ 9\cdot(10^n + 4^{n+2}) = 9\cdot(\mathbb{Z} + \mathbb{N}) = 9\cdot \mathbb{Z}$

$$10^n + 3\cdot 4^{n+2} + 5 = 9\cdot k$$

$$10\cdot 10^n + 4\cdot(3\cdot 4^{n+2}) + 5$$

$$10\cdot(\underbrace{10^n + 3\cdot 4^{n+2}}_{9\cdot k_1} + 5) - 6\cdot(3\cdot 4^{n+2}) - 45$$

$$10\cdot 9\cdot k_1 - 18\cdot 4^{n+2} - 45$$

$$-9\cdot(\underbrace{2\cdot 4^{n+2} + 5}_{n})$$

$$9\cdot(\underbrace{10k_1 - n}_{k_2\in\mathbb{Z}})$$

LÖSUNG

$a_n = a_{n-1} + 2n + 1;\ a_1 = 2.$

a) Intuitiven Form:

$$a_2 = a_1 + 2 \cdot 2 + 1 = 7$$
$$a_3 = a_2 + 2 \cdot 3 + 1 = 14$$
$$a_4 = a_3 + 2 \cdot 4 + 1 = 23$$

$$a_n = 2; 7; 14; 23; 34; 47$$

b) Explizite Darstellung: $a_n = (n+1)^2 - 2$

c) Beweisen Sie die Gültigkeit mittels vollständiger Induktion.

$$a_{n+1} = a_n + 2 \cdot (n+1) + 1 = \left((n+1)+1\right)^2 - 2$$

$$a_n + 2n + 3 = (n^2 + 4n + 4) - 2$$

$$(n+1)^2 - 2 + 2n + 3 = n^2 + 4n + 2$$

$$(n^2 + 2n - 1) + 2n + 3 = n^2 + 4n + 2$$

$$0 = 0$$

$a_n = 2n - 1 \quad , \; n \geq 1$ $\quad \Leftrightarrow \quad$ $b_{n+1} = b_n + 2 \quad , \; b_1 = 1$

explizit $\qquad$ rekursiv

$n = 1 \qquad a_1 = b_1$

$2 \cdot 1 - 1 = 1 \quad \checkmark$

Prämisse $\qquad a_n = b_n = \boxed{2n - 1}$

$n+1 \quad : \quad a_{n+1} = 2 \cdot (n+1) - 1 = 2n + 2 - 1$

$= 2n + 1$

$a_{n+1} = (2n - 1) + 2$

$a_{n+1} = a_n + 2$

$a_n = \frac{3n}{2n+1}$; $n \geq 1$

Monotonie: $a_1 = \frac{3}{2+1} = 1$ $a_2 = \frac{6}{5} = 1\frac{1}{5}$

Behauptung: $a_{n+1} > a_n$

Die Folge ist streng monoton steigend

$n = 1$: $a_2 > a_1$ $1\frac{1}{5} > 1$ ✓

Prämisse: Es gilt $a_{n+1} > a_n$ für $n \geq 1$

$n+1$: $a_{n+2} > a_{n+1}$

$$a_{n+2} > a_{n+1}$$

$$\frac{3(n+2)}{2\cdot(n+2)+1} = \frac{3n+6}{2n+5} > \frac{3\cdot(n+1)}{2\cdot(n+1)+1} = \frac{3n+3}{2n+3} \quad \Big/ \begin{array}{l} \cdot(2n+5) \\ \cdot(2n+3) \end{array}$$

$$(3n+6)(2n+3) > (3n+3)\cdot(2n+5)$$

$$6n^2+12n+9n+18 > 6n^2+6n+15n+15 \quad | -6n^2$$

$$21n+18 > 21n+15 \quad | -21n$$

$$18 > 15 \quad \checkmark$$

<u>Quotient:</u>

$$\frac{a_{n+1}}{a_n} = \frac{\frac{3n+3}{2n+3}}{\frac{3n}{2n+1}} = \frac{3n+3}{2n+3} \cdot \frac{2n+1}{3n}$$

$$\frac{6n^2+9n+3}{6n^2+9n} = \frac{6n^2+9n}{6n^2+9n} + \frac{3}{6n^2+4n} = 1 + (>0) > 1$$

steigt ✓

Differenz

$$a_{n+1} - a_n = \frac{3n+3}{2n+3} - \frac{3n}{2n+1}$$

$$\frac{(3n+3)(2n+1) - 3n\cdot(2n+3)}{(2n+3)(2n+1)}$$

$$\frac{6n^2 + 9n + 3 - 6n^2 - 9n}{(2n+3)(2n+1)} = \frac{3}{\underbrace{(2n+3)}_{>0}\cdot\underbrace{(2n+1)}_{>0}}$$

$$\Rightarrow \frac{3}{>0} > 0$$

$\Rightarrow$ steigend

Schranke: Da (a_n) streng monoton steigend ist,
muss $a_1 = 1$ untere Schranke sein

Behauptung $a_n < 3/2$

$n=1$: $a_1 = 1 < 3/2$ ✓

$n+1$ $a_{n+1} < 3/2$

$\frac{3n+3}{2n+3} < 3/2$ $| \cdot 2 \cdot (2n+3)$

$6n+6 < 6n+9$

$6 < 9$ ✓